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NONLINEAR WAVES ON THE SURFACE OF A FREELY FLOWING VERTICAL LIQUID FILM

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The nonlinear equation describing nonstationary waves on the surface of a freely flowing vertical liquid film has been investigated by a perturbation theory method.

1. The wave flow regimes of thin films on a vertical wall were investigated both experimentally [1-4] and theoretically [4-9] in many studies. Experiment shows that laminar flow of a liquid film is unstable, starting with very small Reynolds numbers. The instability leads to generation of periodic waves on the surface of the film, whose amplitude increases with propagation, and quickly departs from the stationary value. To determine the characteristics of stationary waves, various assumptions on the wave flow regime are usually used in theoretical studies. Thus, in the first problem investigated on wave flow of a vertical liquid film, Kapitza [6] assumed minimal viscous energy dissipation for the wave realized. An assumption was introduced [7] on "optimality" of the wave regime, i.e., minimality (for a given liquid discharge) of the mean film thickness. It was assumed in [4] that only "maximum growth waves" are realized experimentally, for which the amplitude increment is maximum. A problem was subsequently solved [8], where it was taken into account that in the stationary regime the amplitude increment corresponding to the stationary value of the wave number vanishes. At the same time, for all other wave number values the increment must be negative. The use of various assumptions of this nature, such as in [4, 6, 7], often leads to good agreement with experiment, but is, in our opinion, somewhat artificial. It seems to us that the wave characteristics of established flows must be obtained naturally from the solution of the nonstationary nonlinear equation describing the wave formation. In the present paper an attempt was made to solve this problem, using the method of slowly varying parameters, developed in detail by Bogolyubov and Mitropol'skii [10] for nonlinear system oscillations. This method was generalized in [11-16] so as to investigate nonlinear wave processes.

In the region of large Reynols numbers, when $\mathrm{Re}(h_0/\lambda) \gg 1$, the original equation for the film thickness h(x, t) (Fig. 1) is

$$\frac{\partial h}{\partial t}+1.7v_0\frac{\partial h}{\partial x}+2.3v_0\frac{h-h_0}{h_0}\frac{\partial h}{\partial x}-\frac{\sigma h_0}{v_0 d}\frac{\partial^3 h}{\partial x^3}=\frac{3\nu}{h_0^2 v_0}\int\left(\frac{\partial h}{\partial t}+3v_0\frac{\partial h}{\partial x}\right)dx, \tag{1}$$

where $h_0=\sqrt[3]{\frac{3\nu}{g}Q}$; $v_0=\frac{Q}{h_0}$.

The equation given was obtained in [4]. Its linearized variant is also contained in the monograph [9]. For further study of this equation (see [4]), in the right-hand side we replaced the time derivative $\frac{\partial h}{\partial t}$ by $-c\frac{\partial h}{\partial x}\approx -1.7v_0\frac{\partial h}{\partial x}$, and carried out the integration. This

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replacement is not uniformly valid, since the phase velocity c is constant only for short times, when the wave amplitude a is small, and the amplitude dependence of c can be neglected. With increasing a the waves become nonlinear, and c starts depending substantially on amplitude. It is important to turn attention to the fact that Eq. (1) is the Korteweg–de Vries equation with energy pumping, described by the right-hand side. Therefore, the wave evolution and its departure from the stationary regime must be determined by the relation of the terms appearing in the right-hand side of Eq. (1). Even though the quantity $\frac{3\nu}{h_0^2 v_0}\int\left(\frac{\partial h}{\partial t}+3v_0\frac{\partial h}{\partial x}\right)dx$ is small in comparison with the other terms of the equation, the structure of this expression is particularly important for correct treatment.

After transforming to the new unknown function $w = 2.3(h - h_0)/h_0$ and introducing dimensionless variables and parameters

$$t'=\frac{tv_0}{h_0},\quad x'=\frac{x}{h_0},\quad \sigma'=\frac{\sigma}{dv_0^2h_0},\quad \varepsilon=\frac{3\nu}{Q}=\frac{3}{\mathrm{Re}} \tag{2}$$

Eq. (1) transforms to the form

$$\frac{\partial w}{\partial t'}+1.7\frac{\partial w}{\partial x'}+w\frac{\partial w}{\partial x'}-\sigma'\frac{\partial^3 w}{\partial x'^3}=\varepsilon\int\left(\frac{\partial w}{\partial t'}+3\frac{\partial w}{\partial x'}\right)dx'. \tag{3}$$

2. Within our approximation $\mathrm{Re}(h_0/\lambda) \gg 1$ the right-hand side of Eq. (3), proportional to $\varepsilon = 3/\mathrm{Re}$, is small in comparison with each of the term on the left-hand side. Therefore, according to the method of slowly varying coefficients [10] we will assume that the right-hand side of Eq. (3) is a small perturbation influencing its solutions. In the absence of perturbations, i.e., for $\varepsilon = 0$, the solution of Eq. (3) would be the following (see [17, 18]):

$$w_0(x',\ t')=-\rho^2\varphi(z), \tag{4}$$

where

$$\varphi(z)=\mathrm{dn}^2(z,\ s)-E(s)/K(s);$$
$$z=\rho\frac{x'-\eta(t')}{\sqrt{12\sigma'}};\quad \eta(t')=c't'. \tag{5}$$

Here dn(z, s) is the Jacobi elliptic function with modulus s and period 2K(s), corresponding to the wavelength

$$\lambda'=\lambda/h_0=\frac{2}{\rho}\sqrt{12\sigma'}\ K(s), \tag{6}$$

$$c'=\frac{c}{v_0}=1.7+\rho^2\left[\frac{E(s)}{K(s)}-\frac{2-s^2}{3}\right]. \tag{7}$$

The parameter $s\,(0\leqslant s\leqslant 1)$ serves as a measure of the wave nonlinearity. For $s \ll 1$ the elliptic functions tend to the trigonometric functions, while for $s \to 1$ the periodic wave transforms to a single peak. The parameter ρ is related to the wave amplitude. Indeed, if a denotes the difference between the maximum and minimum values of $w_0(x', t')$, then according to the properties of the function dn(z, s)

$$a=\rho^2 s^2. \tag{8}$$

The solution $w_0(x', t')$ was selected by us so that

$$\int_0^{\lambda'} w_0(x',\ t')\,dx'=0. \tag{9}$$

Taking all this into consideration, the general solution of Eq. (3) is sought in form of the following expansion (see [14-16]):

$$w(x', t') = -\rho^2\varphi(z) + \varepsilon w_1(z, t') + \varepsilon^2 w_2(z, t') + \ldots, \tag{10}$$

in which $w_1(z, t')$, $w_2(z, t')$, etc. are periodic functions in the variable z with period 2K(s), and for the stationary regime

$$\int_0^{2K(s)} w_1(z, t')\,dz = \int_0^{2K(s)} w_2(z, t')\,dz = \ldots = 0. \tag{11}$$

The quantities ρ and η are functions of time, and are determined by the following differential equations:

$$\frac{\partial\rho^2}{\partial t'} = \varepsilon A_1(\rho^2) + \varepsilon^2 A_2(\rho^2) + \ldots, \tag{12}$$

$$\frac{\partial\eta}{\partial t'} = c'(\rho^2) + \varepsilon B_1(\rho^2) + \varepsilon^2 B_2(\rho^2) + \ldots \tag{13}$$

Substituting (10) into (3), taking into account Eqs. (12) and (13), we equate the coefficients of identical powers of ε and obtain equations for the functions w_1, w_2, etc. In particular, the equation for $w_1(z, t')$ is

$$\frac{\partial w_1}{\partial t'} - \frac{\rho}{\sqrt{12\sigma'}}\left[(c' - 1.7)\frac{\partial w_1}{\partial z} + \frac{\rho^2}{12}\frac{\partial^3 w_1}{\partial z^3} + \rho^2\frac{\partial}{\partial z}(\varphi w_1)\right] = f[\varphi], \tag{14}$$

where

$$f[\varphi] = -\rho^2\left[1.3 - \left(\frac{E(s)}{K(s)} - \frac{2 - s^2}{3}\right)\right]\varphi + A_1\left(\varphi + \frac{z}{2}\frac{\partial\varphi}{\partial z}\right) - \frac{\rho^3}{\sqrt{12\sigma'}}B_1\frac{\partial\varphi}{\partial z}. \tag{15}$$

The unknown functions $A_1(\rho^2)$ and $B_1(\rho^2)$ introduced by us are determined by the condition of absence in the function $w_1(z, t')$ of secular terms increasing with time. As shown in [13-16], this condition reduces to the orthogonality requirement of the functions φ and $\partial\varphi/\partial z$ in the right-hand side of Eq. (14), i.e., to the satisfaction of the condition

$$\int_0^{2K(s)} f[\varphi]\,\varphi\,dz = \int_0^{2K(s)} f[\varphi]\frac{\partial\varphi}{\partial z}\,dz = 0. \tag{16}$$

From these equalities we find the required functions:

$$A_1(\rho^2) = \beta\rho^2 - \gamma\rho^4, \quad B_1(\rho^2) = \frac{\beta\rho^2 - \gamma\rho^4}{\rho^3}q, \tag{17}$$

where

$$\beta = \frac{1.3\int_0^{2K(s)} \varphi^2 dz}{\int_0^{2K(s)} \varphi^2 dz + \frac{1}{2}\int_0^{2K(s)} z\varphi\frac{\partial\varphi}{\partial z}\,dz};$$

$$\gamma = \frac{\left(\frac{E(s)}{K(s)} - \frac{2-s^2}{3}\right)\int\limits_0^{2K(s)} \varphi^2 dz}{\int\limits_0^{2K(s)} \varphi^2 dz + \frac{1}{2}\int\limits_0^{2K(s)} z\varphi \frac{\partial\varphi}{\partial z}\, dz}; \tag{18}$$

$$q = \frac{\sqrt{3\sigma'}\int\limits_0^{2K(s)} z\left(\frac{\partial\varphi}{\partial z}\right)^2 dz}{\int\limits_0^{2K(s)} \left(\frac{\partial\varphi}{\partial z}\right)^2 dz}.$$

The A_1 and B_1 values found are then substituted into Eqs. (12) and (13), and, restricting ourselves to first-order terms, we obtain

$$\frac{\partial\rho^2}{\partial t'} = \varepsilon\rho^2(\beta - \gamma\rho^2), \tag{19}$$

$$\frac{\partial\eta}{\partial t'} = c'(\rho^2) + \varepsilon\frac{\beta - \gamma\rho^2}{\rho} q. \tag{20}$$

3. The solution of Eq. (19) is the function

$$\rho^2 = \frac{\rho_0^2 \frac{\beta}{\gamma} \exp(\varepsilon\beta t')}{\frac{\beta}{\gamma} + \rho_0^2 [\exp(\varepsilon\beta t') - 1]}. \tag{21}$$

The given relationship makes it possible to investigate the dynamics of amplitude variation of the waves generated. It is seen from Eq. (21) that if the initial value of the wave amplitude $a_0 = \rho_0^2 s^2$ vanishes, then its value $a = \rho^2 s^2$ remains zero at any moment of time, and **there are no waves on the film surface, i.e., the liquid flows laminarly over the vertical wall. This flow is, however, unstable. Since small random runoffs are practically unavoidable in** fluid flow, waves with monotonically increasing amplitudes are automatically excited on the film surface. The amplitude growth is due to the fact that in the right-hand side of Eq. (1) the term $3v_0(\partial h/\partial x)$, referring to a positive energy sink, is larger in absolute value than the term $\partial h/\partial t$, corresponding to a negative sink. With increasing amplitude due to an increase in the phase velocity of the wave (see Eq.(7)) the quantity $\partial h/\partial t$ increases, and for amplitudes equal to

$$a_{st} = \rho_{st}^2 s^2 = \frac{\beta}{\gamma} s^2 = \frac{1.3 s^2}{\frac{E(s)}{K(s)} - \frac{2-s^2}{3}}, \tag{22}$$

the right-hand side of Eq. (1) vanishes. With further amplitude increase the right-hand side of Eq. (1) starts corresponding to a negative energy sink, increasing the wave amplitude toward the stationary value (22).

For periodic waves realized experimentally $s^2 \ll 1$, i.e., the waves are almost sinusoidal. Therefore, expression (22) for a stationary amplitude can be expanded in a series in s:

$$a_{st} = 3.9 s^2\left(1 + \frac{1}{2} s^2 + \ldots\right). \tag{23}$$

According to Eqs. (6), (7), and (20)-(22), the phase velocity and wavelength are:

$$c_{st} = \left[1.7 + \rho_{st}^2\left(\frac{E(s)}{K(s)} - \frac{2-s^2}{3}\right)\right] v_0 = 3v_0 = 2.1\sqrt[3]{\frac{gQ^2}{\nu}}, \tag{24}$$

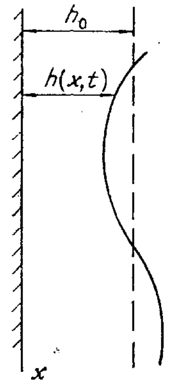

Fig. 1. A flowing-down film.

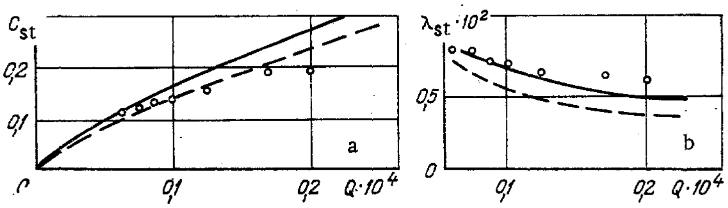

Fig. 2. Stationary values of wave phase velocity (a) and of the wavelength (b) as functions of the liquid flow rate Q for ethyl alcohol; the points are experiment [1], the solid lines are calculated from Eqs. (24) (a) and (25) (b), and the dashed lines are calculated by [6]. c, m/sec; λ_{st}, m; Q, m^2/sec.

$$\lambda_{st} = \frac{2}{\rho_{st}} \sqrt{12\sigma'}\, K(s)\, h_0 = 9.6 \sqrt[3]{\frac{\nu\sigma}{dgQ}} \left(1 + \frac{1}{2} s^4 + \dots\right). \quad (25)$$

Figure 2 shows a comparison of the given theoretical dependences of c_{st} and λ_{st} on the liquid flow rate Q with the experimental dependences [1]. It is seen that the phase velocity of the wave obtained from expression (24) is in good agreement with experiment only at low rates Q; with increasing Q theory provides a faster growth of the phase velocity. The theoretical value of the stationary wavelength λ_{st} is in satisfactory agreement with experiment in almost the whole variation interval of Q.

In conclusion we note that in determining the parameters ρ and η we restricted ourselves to first order terms in Eqs. (12) and (13). Therefore, in expansion (10) for the function w(x', t') it makes no sense to retain terms of first and higher orders, since the errors of Eq. (10) and of the simplified equation

$$w(x', t') = -\rho^2 \varphi(z) \quad (26)$$

are first-order quantities (for a proof see [10]). Consequently, within the first approximation in the parameter ε the wave profile is described by Eq. (26), and it is meaningless to solve Eq. (14).

NOTATION

a, difference between the maximum and minimum $w_0(x', t')$ values; A_i and B_i, quantities introduced in Eqs. (12) and (13); c and c', dimensional and dimensionless phase velocities of the wave; d, liquid density; $f[\varphi]$, function introduced in Eq. (15); g, free-fall acceleration; h(x, t), film thickness; h_0 and v_0, film thickness and flow velocity in the laminar regime; K(s) and E(s), complete elliptic integrals of kinds I and II; Q is the liquid flow rate; Re, Reynolds number; s, Jacobi modulus function; t, t', x, and x', dimensional and dimensionless time and longitudinal coordinate; w(x', t'), relative variation in the film thickness, determined for (2); $w_i(z, t')$, functions introduced in Eq. (10); z, wave phase from Eq. (5); β, γ, q, quantities determined in Eq. (18); ε, smallness parameter introduced in Eq. (2); $\eta(t')$, function introduced in Eqs. (5), (13); λ, λ', dimensional and dimensionless wavlength; ν, kinematic viscosity; ρ, quantity introduced in Eqs. (4), (5); ρ_0, its initial value; σ, σ', dimensional and dimensionless surface tension coefficient; $\varphi(z)$, function introduced in Eqs. (4), (5); a_{st}, c_{st}, γ_{st}, ρ_{st}, values of the parameters a, c, λ, ρ in the stationary regime.

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